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NON-LINEAR CONSTANT-PROFILE WAVES IN A COLD PLASMA UNDER AN APPLIED MAGNETIC FIELD

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Summary

Non-linear coupled differential equations for the 2 transverse components of the local magnetic field have been obtained for a plane wave propagating in a homogeneous cold Vlasov plasma under the influence of an external magnetic field B_o, for any angle $(90^o - \alpha)$ between B_o and the wave vector. There is one exact single valued solution and approximate multiple valued solutions for $\alpha = 90^o$; for $\alpha = 0^o$, the solution is reducible to an integral and is multiple valued. A perturbation method has been used to obtain a restricted class of solutions for intermediate angles. A numerical example has been worked out in detail for a specific value of the field energy density and for 12 different angles; the plasma parameters in this example are those appropriate to the ionosphere of the earth. Other periodic solutions (for which the perturbation solution does not work) may also exist for small values of α, but they have not been considered here.

Author

I. INTRODUCTION

Non-linear plane waves propagating in a fully ionized "cold" collisionless plasma in a magnetic field B_o have previously been studied for the cases of propagation across the magnetic field B_o (magneto-sonic waves), and propagation parallel to the magnetic field (non-linear Alfvén Waves). The former case was considered in detail by Davis, Lüst and Schlüter (Z. Naturforsch. 13 a, 916 (1958)) and the plasma supports partly compressional infinitely periodic waves for restricted values of the transverse electric field. Periodic solutions corresponding to non-linear Alfvén Waves also exist, some of which are incompressible, circularly polarized waves (V.C.A. Ferraro, Proc. Roy. Soc. (London) A233, 310 (1955); D. Montgomery, Phys. Fluids 2, 585 (1959); others are non-polarized compressible waves.

In this study, a more general treatment is given; we look for infinitely periodic waves for any angle (90^o - α) between the wave vector and the applied magnetic field. For such constant-profile waves, there has to be a "wave" coordinate system (moving with a constant velocity V relative to the laboratory frame) in which all quantities are time independent.

We postulate the existence of such a wave frame in part II and, starting from the equations of motion and Maxwell's equations, we derive coupled dimensionless differential equations for the transverse components (i.e., those perpendicular to the direction of propagation) of the magnetic field. The coefficients of these equations contain the angle α as a parameter.

Part III is subdivided into 2 sections:

1) In <u>Section A</u>, a generalized perturbation solution is set up for α Min $< \alpha \leq 90^{\circ}$, where α Min is a small angle. This is based on the fact that in the physical case of interest, one of the coefficients (q) in the equations is much larger than all others for $\alpha > \alpha$ Min. The asymptotic method of Bogoliubov and Mitropolsky is used to avoid the possible appearance of secular (i.e. time proportional) terms; the first-order results are then applied to the cases $\alpha = 90^{\circ}$ and α Min $< \alpha < 90^{\circ}$. For $\alpha = 90^{\circ}$, the solution will give Alfvén Waves as a special case and also more complicated waves propagating in a compressible gas. For α Min $< \alpha < 90^{\circ}$, the gas always behaves compressibly.

2) In <u>Section B</u>, it is shown that for $\alpha = 0^{\circ}$, the solution reduces to an integral, giving linearly polarized waves.

Numerical results are discussed in part IV for plasma parameters corresponding to the earth's ionosphere and the various approximations are discussed in the conclusion.

II. FORMULATION

A. Derivation of the differential equations.

We start with the following equations in the wave frame, in which the operator $\frac{\partial}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial x}$ becomes $\vec{v} \cdot \frac{\partial}{\partial \vec{x}}$:

Maxwell's equations: $$\frac{\partial}{\partial \vec{x}} \cdot \vec{B} = 0 \tag{1}$$

$$\frac{\partial}{\partial \vec{x}} \cdot \vec{E} = 4\pi \sum_{\sigma} e_\sigma n_\sigma \tag{2}$$

$$\frac{\partial}{\partial \vec{x}} \times \vec{E} = 0 \tag{3}$$

$$\frac{\partial}{\partial \vec{x}} \times \vec{B} = \frac{4\pi}{c} \sum_{\sigma} e_\sigma n_\sigma \vec{v}_\sigma \tag{4}$$

Continuity equation: $$\frac{\partial}{\partial \vec{x}} \cdot (n_\sigma \vec{v}_\sigma) = 0 \tag{5}$$

Momentum equation: $$\vec{v}_\sigma \cdot \frac{\partial \vec{v}_\sigma}{\partial x} = \frac{e_\sigma}{m_\sigma} \left(\vec{E} + \frac{\vec{v}_\sigma \times \vec{B}}{c} \right) \tag{6}$$

It will be assumed that we are dealing with a plasma of fully ionized hydrogen; σ is an index denoting the protons or the electrons ($e_p = e$, $e_e = -e$) with densities n_σ and velocities $\vec{v}_\sigma$, and $\vec{E}$, $\vec{B}$ are the local electric and magnetic fields. This choice is appropriate for our later application to the earth's ionosphere. We orient the x - axis parallel to the direction of propagation, so that all variables are functions of x only. The velocity $\vec{v}_\sigma$ has components $(u_\sigma, v_\sigma, w_\sigma)$ along the (x, y, z) axes. Similarly, $\vec{E} = (E_x, E_y, E_z)$, $\vec{B} = (B_x, B_y, B_z)$.

From equation (1), $B_x = B_1 = \text{constant}$ (7)

For the waves we are considering, $E_x = 0\left[\frac{v_\sigma B}{c}\right]$;

substituting this in equation (2),

$$\frac{1}{4\pi}\frac{dE_x}{dx} = e\,(n_p - n_e) = e\,\Delta n = 0\left[\frac{v_\sigma B}{4\pi c L}\right];$$

$$\text{or } \frac{\Delta n}{n} \simeq 0\left[\frac{v_\sigma B}{4\pi e c L n}\left(\frac{m_p B}{m_p B}\right)\right] = 0\left[\left(\frac{V_A}{c}\right)^2 \frac{R_e}{L}\right] << 1,$$

where:

L = gradient scale

V_A = Alfvén speed

R_e = cyclotron radius

Therefore, $n_p(x) \simeq n_e(x) = N$ (8)

It follows from a formal expansion of (3) that

$E_y = E_2 = \text{constant}$ (9)

$E_z = E_3 = \text{constant}$ (10)

We now proceed to obtain 2 equations for the variables B_y and B_z from the remaining equations (4), (5), (6). It is also useful at this stage to change from e. s. to e.m. units by means of the transformation $e \to ce$, $\vec{E} \to \frac{\vec{E}}{c}$, $\vec{B} \to \vec{B}$.

Reducing equation (4) to 3 scalar equations and substituting (8) in the first one, we obtain:

$$u_p = u_e = U \tag{11}$$

We then define a density flux F for both types of particles:

$$F = F_p = F_e = n_p\, u_p = n_e\, u_e = NU \tag{12}$$

The other 2 scalar equations from (4) give in the same way:

$$\frac{dB_z}{dx} = -4\,\pi\, e\, N\, (v_p - v_e) \tag{13}$$

$$\frac{dB_y}{dv} = 4\,\pi\, e\, N\, (w_p - w_e) \tag{14}$$

We also reduce the momentum equation (6) to 3 scalar equations in u_σ, v_σ, w_σ by formal expansion. Substituting (13), (14) in the first one, summing over σ, defining $m_t = m_p + m_e$ and $\bar{F} = m_t NU$ (Total mass flux in the direction of wave propagation), we obtain:

$$\bar{F}\frac{dU}{dx} = -\frac{1}{4\pi}\left[B_y\left(\frac{dB_y}{dx}\right) + B_z\left(\frac{dB_z}{dx}\right)\right]$$

This may be integrated directly to give:

$$\bar{F}U + \frac{1}{8\pi}\left[B_y^{\,2} + B_z^{\,2}\right] = \Pi_o = \text{constant}, \tag{15}$$

a familiar conservation equation.

The second and third scalar equations derived from (6) also lead to corresponding conservation equations along the 2 transverse directions. With the substitution of (7), (13), (14), they take the form:

$$\bar{F}\left[m_p v_p + m_e v_e\right] = \frac{m_t\, B_1\, B_y}{4\,\pi} + \Pi_2 \tag{16}$$

$$\bar{F}\left[m_p w_p + m_e w_e\right] = \frac{m_t\, B_1\, B_z}{4\,\pi} + \Pi_3 \tag{17}$$

where Π_2 and Π_3 are constants.

We now eliminate E_x and N as variables. We return to the scalar equations obtained from (6), but do not sum over σ, obtaining expressions for

$$u_\sigma \frac{du_\sigma}{dx}, \quad u_\sigma \frac{dv_\sigma}{dx}, \quad u_\sigma \frac{dw_\sigma}{dx}$$

for both protons and electrons. Using $u_p = u_e = U$ from (11), and subtracting the first 2 equations gives:

$$E_x = \frac{1}{m_t}\left\{B_y\,(m_e\,w_p + m_p\,w_e) - B_z(m_e\,v_p + m_p\,v_e)\right\} \tag{18}$$

Next, we substitute $N = \frac{F}{U}$ from (12) into (13), (14) and differentiate the resulting equations with respect to time:

$$\ddot{B}_z = -4\pi eF\,(\dot{v}_p - \dot{v}_e) \text{ and } \ddot{B}_y = 4\pi eF\,(\dot{w}_p - \dot{w}_e)\ ,$$

where $\dot{v}_p$, $\dot{v}_e$, $\dot{w}_p$ and $\dot{w}_e$ may be eliminated from the last 4 scalar equations from (6). This yields

$$\ddot{B}_z = -4\pi e^2 F\left[\frac{1}{m_p}\left\{E_2 + (w_p B_1 - UB_z)\right\} + \frac{1}{m_e}\left\{E_2 + (w_e B_1 - UB_z)\right\}\right] \tag{19}$$

$$\ddot{B}_y = 4\pi e^2 F\left[\frac{1}{m_p}\left\{E_3 + (UB_y - B_1 v_p)\right\} + \frac{1}{m_e}\left\{E_3 + (UB_y - B_1 v_e)\right\}\right] \tag{20}$$

Finally, we must eliminate the variables u, v_p, v_e, w_p, w_e from equations (19) and (20); U is directly obtained from (15), and the transverse velocity components obtained from (13), (14), (16), (17) after a considerable amount of algebra. After substitution of these quantities, equations(19) and (20) become:

$$\ddot{B}_y = \frac{4\pi e^2}{m_p m_e}\left[\left(\bar{F}E_3 - \frac{\Pi_2 B_1}{m_t}\right) - \frac{B_1^{\,2}B_y}{4\pi} + \left\{\Pi_o - \frac{(B_y^{\,2}+B_z^{\,2})}{8\pi}\right\}\left\{B_y + \frac{(m_e-m_p)}{4\pi e\bar{F}} B_1\left(\frac{dB_z}{dx}\right)\right\}\right]$$

$$\ddot{B}_z = \frac{-4\pi e^2}{m_p m_e}\left[\left(\bar{F}E_2 + \frac{\Pi_3 B_1}{m_t}\right) + \frac{B_1^{\,2}B_z}{4\pi} + \left\{\Pi_o - \frac{(B_y^{\,2}+B_z^{\,2})}{8\pi}\right\}\left\{\frac{(m_e-m_p)}{4\pi e\bar{F}} B_1\left(\frac{dB_y}{dx}\right) - B_z\right\}\right]$$

It is convenient to rewrite the above equations in a dimensionless form by defining some new dimensionless variables and constants:

$$B_1 = b_1\sqrt{8\pi\Pi_o} \qquad B_y = b_y\sqrt{8\pi\Pi_o} \qquad B_z = b_z\sqrt{8\pi\Pi_o}$$

$$E_x = \frac{\epsilon_x\Pi_o}{\bar{F}}\sqrt{8\pi\Pi_o} \qquad E_2 = \frac{\epsilon_2\Pi_o}{\bar{F}}\sqrt{8\pi\Pi_o} \qquad E_3 = \frac{\epsilon_3\Pi_o}{\bar{F}}\sqrt{8\pi\Pi_o}$$

$$\Omega_o^{\,2} = \frac{4\pi e^2\Pi_o}{m_p m_e} \qquad t = \frac{\tau}{\Omega_o} \qquad x = \frac{(m_e-m_p)\,\xi\sqrt{8\pi\Pi_o}}{4\pi e\bar{F}}$$

$$\frac{\Pi_2}{\Pi_o m_t} = k_2 \qquad \frac{\Pi_3}{\Pi_o m_t} = k_3 \qquad \zeta = \frac{\xi}{b_1}$$

$$m = \epsilon_3 - k_2 b_1 \qquad n = -\epsilon_2 - k_3 b_1 \qquad p = -2\,b_1^{\,2}$$

$$q = \frac{b_1\sqrt{2}\,(m_e-m_p)}{\sqrt{m_p m_e}}$$

The resulting equations further become homogeneous in the derivatives by the formal substitution of $\frac{U\,d}{dx} = \frac{d}{dt}$ and (15). We then obtain:

$$\frac{d^2 b_y}{d\tau^2} = m + pb_y + b_y \left\{1 - (b_y{}^2 + b_z{}^2)\right\} + q \frac{db_z}{d\tau} \tag{21}$$

$$\frac{d^2 b_z}{d\tau^2} = n + pb_z + b_z \left\{1 - (b_y{}^2 + b_z{}^2)\right\} - q \frac{db_y}{d\tau} \tag{22}$$

The problem is to solve (21) and (22) for any given angle α. However this cannot be done with the existing equations since b_y and b_z (we do not need to worry about the constant $b_x = b_1$) are the dimensionless components of the total local instantaneous magnetic field, and this last vector will generally make an angle γ with the plane waves where, of course, γ is itself some unknown function of position.

This difficulty is easily resolved by introducing 2 new variables β_y and β_z to replace b_y and b_z (refer to Figure 1):

$$b_x = b_1 = b_o \sin \theta_o \cos \Phi_o$$

$$b_y = b_2 + \beta_y = b_o \sin \theta_o \sin \Phi_o + \beta_y$$

$$b_z = b_3 + \beta_z = b_o \cos \theta_o + \beta_z$$

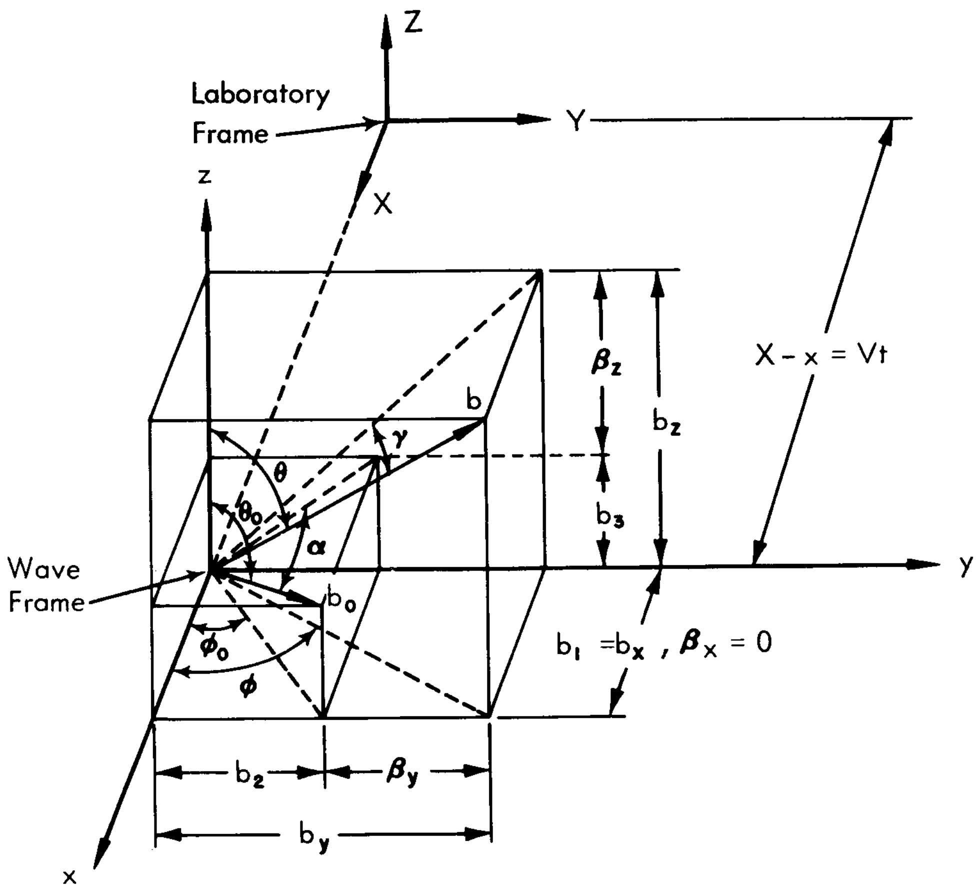

Figure 1. Components of b and b_o in the Wave Frame

So far, only the position of the x axis in space has been determined; the orientation of the y, z axes is arbitrary. However, the equations for β_y and β_z have their most symmetric form when $b_2 = b_3$. In that case,

$$b_x = b_1 = b_o \sin \alpha$$

$$b_y = b_2 + \beta_y = \frac{b_o}{\sqrt{2}} \cos \alpha + \beta_y$$

$$b_z = b_3 + \beta_z = \frac{b_o}{\sqrt{2}} \cos \alpha + \beta_z$$

It is also useful to substitute $T = q\tau$, $(q^2)^{-1} = \epsilon$. After these changes, (21), (22) take the form:

$$\frac{d^2\beta_y}{dT^2} - \frac{d\beta_z}{dT} = \epsilon \left[A + B\beta_y + C\beta_y^{\,2} - \beta_y^{\,3} + D\beta_z + E\beta_y\beta_z + F\beta_z^{\,2} - \beta_y\beta_z^{\,2} \right] \quad (23)$$

$$\frac{d^2\beta_z}{dT^2} + \frac{d\beta_y}{dT} = \epsilon \left[G + B\beta_z + C\beta_z^{\,2} - \beta_z^{\,3} + D\beta_y + E\beta_y\beta_z + F\beta_y^{\,2} - \beta_y^{\,2}\beta_z \right] \quad (24)$$

where the coefficients A, B, . . . G, q (or ϵ) are constants with the angle α as a parameter. They are:

$$A = \frac{1}{\Pi_o\sqrt{8\pi\Pi_o}}\left[(m_e + m_p)\ NUE_s - \frac{\Pi_2 B_o \sin\alpha}{(m_e + m_p)} + \frac{B_o \cos\alpha}{\sqrt{2}}\left(\Pi_o - \frac{B_o^2}{8\pi} - \frac{B_o^2 \sin^2\alpha}{8\pi}\right)\right]$$

$$G = \frac{1}{\Pi_o\sqrt{8\pi\Pi_o}}\left[-(m_e + m_p)\ NUE_2 - \frac{\Pi_3 B_o \sin\alpha}{(m_e + m_p)} + \frac{B_o \cos\alpha}{\sqrt{2}}\left(\Pi_o - \frac{B_o^2}{8\pi} - \frac{B_o^2 \sin^2\alpha}{8\pi}\right)\right]$$

$$B = \left[1 - \frac{B_o^2}{4\pi\Pi_o}\right] = \left[1 - \frac{1}{k^2}\right] \qquad C = -\frac{3B_o \cos\alpha}{4\sqrt{\pi\Pi_o}} = -\frac{3\cos\alpha}{2k}$$

$$D = -\frac{B_o^2 \cos^2\alpha}{8\pi\Pi_o} = -\frac{\cos^2\alpha}{2k^2} \qquad E = -\frac{B_o \cos\alpha}{2\sqrt{\pi\Pi_o}} = -\frac{\cos\alpha}{k}$$

$$F = -\frac{B_o \cos\alpha}{4\sqrt{\pi\Pi_o}} = -\frac{\cos\alpha}{2k}$$

$$q = \frac{1}{\sqrt{\epsilon}} = \frac{B_o \sin\alpha\,(m_e - m_p)}{2\sqrt{\pi\Pi_o}\sqrt{m_p m_e}} = \frac{\sin\alpha\,(m_e - m_p)}{k\sqrt{m_p m_e}}$$

B. Evaluation of the coefficients in the equations.

In order to solve the coupled equations (23) and (24) in a given situation, their coefficients must be known. These quantities depend on α, B_o (which we assume to be 0.39 gauss corresponding to the mean geomagnetic field at an altitude of 200 km. above the earth's surface), Π_o, Π_2, Π_s, E_2 and E_s. The last 5 constants are so far arbitrary, since they were defined as integration constants. We now briefly outline how to obtain the order of magnitude of these constants.

1. From (15) and the definitions of α, $\bar{F}$,

$$\Pi_o = m_t NU^2 + \frac{1}{8\pi}[B_y{}^2 + B_z{}^2]$$

$$= m_t NU^2 + \frac{1}{8\pi}\left[\left(\frac{B_o}{\sqrt{2}} \cos\alpha + B_y\right)^2 + \left(\frac{B_o}{\sqrt{2}} \cos\alpha + B_z\right)^2\right]$$

$$\text{where } B_y = \sqrt{8\pi\Pi_o}\,\beta_y,\ B_z = \sqrt{8\pi\Pi_o}\,\beta_z$$

We require that $m_t NU^2 \geq 0$ so that the characteristics do not cross themselves. For $\alpha = 90^o$, it will later be shown (refer to III.A) that there is an exact solution:

$$B_y = \sqrt{8\pi\Pi_o}\,\sqrt{B}\,\sin\Psi,\ B_z = \sqrt{8\pi\Pi_o}\,\sqrt{B}\,\cos\Psi, \text{ where } B \geq 0$$

With $B = 1 - \dfrac{B_o{}^2}{4\pi\Pi_o}$ (from Page 12), we obtain after substitution:

$$\Pi_o = \frac{k^2 B_o^{\,2}}{4\pi}\ , \text{ where } 1 \le k < \infty$$

More generally, we also take $\Pi_o = \dfrac{k^2 B_o^{\,2}}{4\pi}$ and adjust the amplitude of B_y, B_z to keep

$$m_t N U^2 \ge 0.$$

2. Substituting $B_y = \dfrac{B_o}{\sqrt{2}} \cos \alpha + B_Y$ in (16) and rewriting this equation, we obtain:

$$K_1 = K_2 \left[m_p v_p + m_e v_e \right] - K_s \ ,$$

where K_1, K_2 and K_s are constants. We take the average of this equation over 1 cycle, noting that

$$\bar{B}_Y = 0,\ \bar{K} = K,\ \text{etc} \ldots,\ m_t \simeq m_p$$

The same arguments are valid for Π_s by symmetry, and:

$$\Pi_2 = \Pi_s = \frac{m_t B_o^{\,2}}{4\pi} \left[K - \frac{\sin \alpha \cos \alpha}{\sqrt{2}} \right], \text{ where } 0 \le K < \infty$$

3. From the symmetrical choice of the y - z axes

relative to $\vec{B}_o$,

$$E_2 = -E_s = \frac{E}{\sqrt{2}}$$

We substitute this and the results obtained above for Π_o, Π_2, and Π_s in the expressions for A, G given previously. After some algebra, we get:

$$A = G = \frac{1}{k^3\sqrt{2}} \left[K\,(1 - \sin\alpha) + \frac{\cos\alpha}{2\sqrt{2}}\,(2k^2 - \cos^2\alpha) \right]$$

This implies that $A = G = 0$ for $\alpha = 90^o$ and we will return to this point in part III. A.

III. SOLUTION

A. Solution for α Min $< \alpha \leq 90^\circ$.

We rewrite (23), (24) in the simplified form:

$$\frac{d^2\beta_y}{dT^2} - \frac{d\beta_z}{dT} = \epsilon\Lambda \tag{23'}$$

$$\frac{d^2\beta_z}{dT^2} + \frac{d\beta_y}{dT} = \epsilon\Gamma \tag{24'}$$

where $\Lambda = \Lambda\ (\beta_y, \beta_z)$, $\Gamma = \Gamma\ (\beta_y, \beta_z)$, and ϵ is small. The zeroth-order solution ($\epsilon = 0$) of these equations is trivial:

$$\beta_y^{(0)} = a \sin \Psi$$

$$\beta_z^{(0)} = a \cos \Psi$$

where $\frac{da}{dT} = 0$, $\frac{d\Psi}{dT} = 1$, $\Psi = T + \theta$

We now differentiate (23') and (24') once, and rearrange the results to uncouple the dominating terms:

$$\frac{d^3\beta_y}{dT^3} + \frac{d\beta_y}{dT} = \epsilon\left[\Gamma + \frac{d\Lambda}{dT}\right] = \epsilon M \tag{25}$$

$$\frac{d^3\beta_z}{dT^3} + \frac{d\beta_z}{dT} = \epsilon\left[\frac{d\Gamma}{dT} - \Lambda\right] = \epsilon N \tag{26}$$

It is now possible to write a perturbation expansion when $\epsilon \neq 0$, using the Bogoliubov-Mitropolsky formalism*:

$$\beta_y = a \sin \Psi + \epsilon\, \beta_y^{(1)}(a, \Psi) + \ldots$$

$$\beta_z = a \cos \Psi + \epsilon\, \beta_z^{(1)}(a, \Psi) + \ldots$$

where:

$$\frac{da}{dT} = \epsilon\, A_1\,(a) + \ldots$$

$$\frac{d\Psi}{dT} = 1 + \epsilon B_1\,(a) + \ldots$$

Using these expressions, the left-hand sides of equations (25), (26) may now be written after some algebra, to order ϵ:

$$\frac{d^3\beta_y}{dT^3} + \frac{d\beta_y}{dT} = \epsilon\left[\frac{\partial^3\beta_y^{(1)}}{\partial\Psi^3} + \frac{\partial\beta_y^{(1)}}{\partial\Psi} - 2\, aB_1 \cos\Psi - 2A_1 \sin\Psi\right]$$

---

* N. N. Bogoliubov and Y. A. Mitropolsky, Asymptotic Methods in the theory of Non-Linear Oscillations, 1961, Gordon and Breach Science Publishers, New York.

$$\frac{d^3\beta_z}{dT^3} + \frac{d\beta_z}{dT} = \epsilon\left[\frac{\partial^3\beta_z^{(1)}}{\partial\Psi^3} + \frac{\partial\beta_z^{(1)}}{\partial\Psi} + 2\,a\,B_1 \sin\Psi - 2A_1 \cos\Psi\right]$$

These results should be equated to the corresponding quantities in the right hand sides:

$$\frac{\partial^3\beta_y^{(1)}}{\partial\Psi^3} + \frac{\partial\beta_y^{(1)}}{\partial\Psi} = f_o\,(a,\Psi) + 2A_1 \sin\Psi + 2aB_1 \cos\Psi \tag{27}$$

$$\frac{\partial^3\beta_z^{(1)}}{\partial\Psi^3} + \frac{\partial\beta_z^{(1)}}{\partial\Psi} = l_o\,(a,\Psi) + 2A_1 \cos\Psi - 2aB_1 \sin\Psi \tag{28}$$

where f_o and l_o are the zeroth-order expansions of M and N. We will solve (27), (28) for $\beta_y^{(1)}, \beta_z^{(1)}$ by Fourier expanding f_o, l_o as well as $\beta_y^{(1)}, \beta_z^{(1)}$:

$$f_o\,(a,\Psi) = g_o\,(a) + \sum_{n=1}^{\infty}\left\{g_n\,(a)\cos n\Psi + h_n\,(a)\sin n\Psi\right\}$$

$$l_o\,(a,\Psi) = p_o(a) + \sum_{n=1}^{\infty}\left\{p_n(a)\cos n\Psi + q_n\,(a)\sin n\Psi\right\}$$

$$\beta_y^{(1)}(a,\Psi) = v_o\,(a) + \sum_{n=1}^{\infty}\left\{v_n\,(a)\cos n\Psi + w_n\,(a)\sin n\Psi\right\}$$

$$\beta_z^{(1)}(a, \psi) = u_o(a) + \sum_{n=1}^{\infty} \left\{ u_n(a) \cos n\psi + r_n(a) \sin n\psi \right\}$$

We substitute these series in (27), (28) and equate coefficients of identical harmonics to get:

$$g_o(a) = p_o(a) = 0$$

$$v_1(a) = \omega_1(a) = u_1(a) = r_1(a) = 0$$

for $n = 2, 3, \ldots$:

$$v_n(a) = \frac{g_n(a)}{n(n^2-1)} \qquad \omega_n(a) = \frac{-h_n(a)}{n(n^2-1)}$$

$$u_n(a) = \frac{q_n(a)}{n(n^2-1)} \qquad r_n(a) = \frac{-p_n(a)}{n(n^2-1)}$$

With the conditions:

$$g_1(a) + 2a\, B_1 = 0 \qquad h_1(a) + 2A_1 = 0$$

$$q_1(a) - 2a\, B_1 = 0 \qquad p_1(a) + 2A_1 = 0$$

We note that $v_o(a)$ and $u_o(a)$ remain undetermined so far, because they have disappeared in the differentiations. We will return to this point in the next section.

We now apply these general results to the two cases of interest:

I) $\underline{\alpha = 90^{\circ}}$

Then $C = D = E = F = 0$. After substitution of the zeroth order results and some simplifications, f_o and l_o finally become:

$$f_o\ (a, \Psi) = \left[A + 2a\ (B-a^2)\ \cos \Psi\right]$$

$$l_o\ (a, \Psi) = -\left[A + 2a\ (B-a^2)\ \sin \Psi\right]$$

We identify these terms with those of the Fourier expansions and then apply the general results on page 19 to get:

$$A = g_o\ (a) = -\ p_o\ (a) = 0$$

$$v_n = \omega_n = u_n = r_n = 0 \text{ for } n = 1, 2, 3, \ldots$$

$$A_1 = 0 \qquad\qquad B_1 = a^2 - B$$

However, $v_o(a)$ and $u_o(a)$ are still undetermined so far. Since $A_1 = 0$, the only effect of $v_o(a)$ and $u_o(a)$ is to add small constant terms to $\beta_y^{(1)}$ and $\beta_z^{(1)}$; from their definitions, the time average of β_y, β_z over a cycle must be zero. Therefore $v_o(a) = u_o(a) = 0$.

We then have, to first order,

$$\beta_y = a \sin \Psi \qquad \beta_z = a \cos \Psi$$

$$\frac{da}{dT} = 0 \qquad \frac{d\Psi}{dT} = 1 + \epsilon\,(a^2 - B)$$

II) $\underline{\alpha \text{ Min} < \alpha < 90^\circ}$:

We first substitute $C = 3F$, $D = -2F^2$, $E = 2F$ and evaluate f_o, l_o as before:

$$f_o(a, \Psi) = \left[(A + 2F a^2) + 2a(B - a^2) \cos \Psi + 3Fa^2 \cos 2\Psi + 3Fa^2 \sin 2\Psi\right]$$

$$l_o(a, \Psi) = -\left[(A + 2F a^2) + 2a(B - a^2) \sin \Psi - 3Fa^2 \cos 2\Psi + 3Fa^2 \sin 2\Psi\right]$$

Proceeding as in the case $\alpha = 90^\circ$, we obtain:

$$A + 2Fa^2 = g_o(a) = -p_o(a) = 0$$

$$v_2 = -\omega_2 = -u_2 = -r_2 = \frac{Fa^2}{2}$$

$$v_n = \omega_n = u_n = r_n = 0, \; n \neq 2$$

$$A_1 = 0 \qquad\qquad B_1 = a^2 - B$$

which gives the first order solution:

$$\beta_y = a \sin \Psi + \frac{\epsilon\ Fa^2}{2}\ (\cos 2\Psi - \sin 2\Psi) \qquad \frac{da}{dT} = 0$$

$$\beta_z = a \cos \Psi - \frac{\epsilon\ Fa^2}{2}\ (\cos 2\Psi + \sin 2\Psi) \qquad \frac{d\Psi}{dT} = 1 + \epsilon\ (a^2 - B)$$

However, it can be verified that the above results do not quite satisfy the original equations (23'), (24') to order ϵ, although they satisfy exactly the derived equations (25), (26) from which they were obtained: this is because we lost some information in differentiating (23'), (24') to obtain (25), (26) This can be remedied by adding the terms ϵF^2 a $\cos \psi$ and ϵF^2 a $\sin \psi$ to β_y and β_z respectively. The modified solution will now satisfy (23'), (24') as well as (25), (26) to order ϵ; it also reduces exactly to the solution found previously for $\alpha = 90^\circ$.

The final first order solution then takes the form, for α Min $< \alpha \leq 90^\circ$:

$$\beta_y = a \sin \Psi + \frac{\epsilon Fa^2}{2} (\cos 2\Psi - \sin 2\Psi) + \epsilon\ F^2 a \cos \Psi \qquad (29)$$

$$\beta_z = a \cos \Psi - \frac{\epsilon Fa^2}{2} (\cos 2\Psi + \sin 2\Psi) + \epsilon\ F^2 a \sin \Psi \qquad (30)$$

where $\frac{da}{dT} = 0 \qquad \frac{d\Psi}{dT} = 1 + \epsilon\,(a^2 - B) \qquad A + 2Fa^2 = 0$

Let us first consider the case $\alpha = 90^\circ$ in more detail, i.e. $F = 0$. An interesting special case is to choose $a^2 = B$ (No frequency shift). The first order solution then becomes the exact solution, giving circularly polarized waves propagating in an incompressible medium (Alfvén Waves). This agrees with the conclusions of Ferraro. After some algebraic manipulations, we can then obtain $B_y = B_y\ (x)$, $B_z = B_z\ (x)$ exactly:

$$B_x = B_o$$

$$B_y = B_o \sqrt{2\ (k^2-1)} \quad \sin \left[\frac{ex\ (m_e - m_p)\ \sqrt{4\pi\ (m_e + m_p)\ N}}{m_e\, m_p} \right]$$

$$B_z = B_o \sqrt{2\ (k^2-1)} \quad \cos \left[\frac{ex\ (m_e - m_p)\ \sqrt{4\pi\ (m_e + m_p)\ N}}{m_e\, m_p} \right]$$

The wavelength λ is therefore given by:

$$\lambda = \left[\frac{2\pi\, m_p\, m_e}{e\ (m_p - m_e)\ \sqrt{4\pi\ N\ (m_e + m_p)}} \right] \sim \frac{V_A}{\Omega_e}$$

where $\Omega_e = \frac{eB_o}{m_e}$ is the electron cyclotron frequency. It may also be verified from (15) that

$$U = \text{constant} = \frac{B_o}{\sqrt{4\pi N\,(m_e + m_p)}} = V_A \text{ (Alfvén Speed)}$$

For $a^2 \neq B$, the waves are more complicated and propagate in a compressible medium. This is always the case for intermediate angles.

We observe that the condition $A + 2Fa^2 = 0$ always gives a relation between the transverse electric field E' and the angle α, for a given field energy density and the amplitude a.

Finally, it must be stressed that this solution is only valid for α Min $< \alpha \leq 90^\circ$. The minimum angle α Min depends on the field energy density and the type of ions present. (Refer to the definition of ϵ on Page 12). This is also discussed in part IV.

B. <u>Solution for $\alpha = 0^\circ$ (Magnetosonic Waves).</u>

Then $q = 0$ and equations (21), (22) reduce to:

$$\frac{d^2\beta_y}{d\tau^2} = A + B\beta_y + C\beta_y^2 - \beta_y^3 + D\beta_z + E\beta_y\beta_z + F\beta_z^2 - \beta_y\beta_z^2 \tag{21'}$$

$$\frac{d^2\beta_z}{d\tau^2} = A + B\beta_z + C\beta_z^2 - \beta_z^3 + D\beta_y + E\beta_y\beta_z + F\beta_y^2 - \beta_y^2\beta_z \tag{22'}$$

A first possibility is that $\vec{B}$ may be linearly polarized along $\vec{B}_o$; then $\beta_y = \beta_z = \beta$ at all times and equations (21'), (22') become identical:

$$\frac{d^2\beta_y}{d\tau^2} = \frac{d^2\beta_z}{d\tau^2} = \frac{d^2\beta}{d\tau^2} = A + \beta\ (B + D) + \beta^2\ (C + E + F) - 2\beta^3$$

From intuition, we expect to obtain infinitely periodic solutions in β for restricted values of A, B, D, . . since the last equation has the form of a modified simple-harmonic oscillator equation.

A more formal way of seeing this is to transform the differential equation to an integral:*

$$\frac{1}{2}\left(\frac{d\beta}{d\tau}\right)^2 = A\beta + \frac{\beta^2}{2}\ (B + D) + \frac{\beta^3}{3}\ (C + E + F) - \frac{\beta^4}{2} + \Pi_4\ ,$$

where Π_4 is a constant. It follows that the formal solution may be written as:

$$\int_{\tau_1}^{\tau_2} d\tau = \frac{1}{\sqrt{2}} \int_{\beta(\tau_1)}^{\beta(\tau_2)} \frac{d\beta}{\sqrt{A\beta + \frac{\beta^2}{2}\ (B + D) + \frac{\beta^3}{3}\ (C + E + F) - \frac{\beta^4}{2} + \Pi_4}}$$

The allowed limits on A for which a periodic solution will exist may be obtained by solving

---

*This approach is similar to a treatment developed by David, Lüst and Schlüter, except that we use β as the variable instead of the total field intensity b: this adds a cubic term to the "potential".

$$A + \beta\,(B + D) + \beta^2\,(C + E + F) - 2\beta^3 = 0$$

so as to have real roots. After substituting $C + E + F = -\frac{3}{k}$, $B + D = \frac{2k^2 - 3}{2k^2}$, we obtain:

$$\frac{2k^2 - 1}{4k^3} - \frac{\sqrt{2}}{3\sqrt{3}} < A < \frac{2k^2 - 1}{4k^3}$$

These limits are consistent with those obtained for the electric field E' using the total field b instead of β. If we return to Page 13 and investigate the no-looping condition for $\alpha = 0^{\circ}$, this condition may be expressed as follows:

$$-\left(\frac{k\sqrt{2} + 1}{2k}\right) < \beta < \left(\frac{k\sqrt{2} - 1}{2k}\right)$$

Since E' (or A) may have any value between 2 finite limits, the wavelength λ is not uniquely determined in this case, but will depend upon the parameter A and the initial conditionsin a rather complicated way. There are 2 interesting special cases in which the wavelength becomes infinite:

a) When $E' = 0$ ($A = A\ \mathrm{Max} = \frac{2k^2 - 1}{4k^3}$), $\lambda \to \infty$ if we set the initial conditions so that

$$\beta = \left(\frac{k\sqrt{2} - 1}{2k}\right) \quad \text{or} \quad \beta = -\left(\frac{k\sqrt{2} + 1}{2k}\right), \ \frac{d\beta}{d\tau} = 0$$

b) When $E' = \frac{2}{3\sqrt{3}} \; \frac{\Pi_0 \sqrt{8\pi\Pi_0}}{\bar{F}} \left(A = A\ Min = \frac{2k^2 - 1}{4k^3} \; \frac{-\sqrt{2}}{3\sqrt{3}} \right)$,

$\lambda \rightarrow \infty$ if

$$\beta = \left(- \frac{1}{2k} + \frac{1}{\sqrt{6}} \right), \; \frac{d\beta}{d\tau} = 0$$

Physically this means that we do not really have waves any more, all physical variables are constant in space and time:

$$\frac{d\beta}{d\tau} = \frac{d^2\beta}{d\tau^2} = 0$$

IV. NUMERICAL RESULTS

A number of arbitrary constants (Π_0, Π_2, . .) were introduced during the derivation of the original differential equations, and the total number of modes is therefore very great; however, we simply want to illustrate the typical behavior of the waves, so we first decided on a single value of Π_0 or $k = \sqrt{1.005}$*, and $a^2 = B$. The constant $\bar{\bar{F}}$ is determined from the exact result for $\alpha = 90^\circ$ on Pages 23 - 24; all other coefficients (except A) in the differential equations were evaluated for 12 different angles using the formulas on Pages 12 - 15 while A itself was determined from the condition $A + 2FB = 0$ found on Page 23 for α Min $< \alpha \leq 90^\circ$; for $\alpha = 0^\circ$, any value may be taken between the limits given on Page 25 - 26. In order to have a finite wavelength, we took the typical value $A = 0$.

The exact equations (23), (24) were then solved numerically for 12 values of α on an IBM 7094 computer using initial values calculated from the perturbation results (29), (30).

The wavelength λ was determined to a good accuracy for each angle by inspection of the tabular results for B_y and B_z against x, noting at which points the field quantities returned to their initial values. For $\alpha = 90^\circ$, this method gave $\lambda \simeq 122$ cm., which

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* This corresponds to (B_y) Max = (B_z) Max = $0.1\ B_o$ for $\alpha = 90^\circ$.

agrees quite well with the analytical result (123 cm.) calculated from the formula on Page 23.

The field quantities are plotted against x (for $o \leq x \leq \lambda$) for each angle on Pages 33 - 34, and a graph of λ against α is shown on Page 35; the results for $o \leq \alpha < 2.5^{\circ}$ are interpolated since the accuracy of the perturbation expansion is not good for very small angles.

The order of magnitude of the minimum angle α Min at which the perturbation theory will break down may be obtained approximately by using the ratio test: $\epsilon^2 < 1$. This gives α Min $= O(1^{\circ})$ for the parameters chosen in the example.

V. CONCLUSION

The problem of non-linear plane waves propagating in a fully ionized hydrogen plasma under an applied magnetic field B_o at an angle $(90^o - \alpha)$ between the wave vector and B_o has been divided into 2 parts: if the angle α is larger than a certain critical angle α Min, a first-order perturbation solution was obtained for α Min $< \alpha \leq 90^o$; for $\alpha = 0^o$, the solution is reducible to an integral and is multiply valued; for $0 < \alpha \leq \alpha$ Min, the problem remains unsolved. Fortunately, α Min is quite small here $(O(1^o))$, so that the perturbation result gives a solution over most values of α.

Because of local fluctuations in B_o and N, the "infinitely periodic waves" described here will actually only propagate undistorted in regions of the ionosphere where these quantities are constant. Finally, the assumption of a "fully ionized hydrogen plasma" is only an idealization to keep the equations tractable.

VI. ACKNOWLEDGEMENT

I am indebted to Dr. D. A. Tidman for useful discussions and initially suggesting this problem, and to Dr. D. C. Montgomery who supplied a copy of the important paper of Davis, Lüst and Schlüter. I would also like to thank Dr. Gilbert Mead for the information on the geomagnetic field. The programming of the differential equations for the numerical solution was skilfully done by Mr. Robert F. Baxter.

TABLE I.

Numerical values of coefficients in the differential equations and initial values of the variables.*

α	0°	2.5°	5.0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
A = G	0.000000**	0.495802x10^{-2}	0.494383x10^{-2}	0.488735x10^{-2}	0.466343x10^{-2}	0.429786x10^{-2}	0.380165x10^{-2}	0.319000x10^{-2}	0.248136x10^{-2}	0.169735x10^{-2}	0.861778x10^{-3}	0.000000
B	0.497512x10^{-2}	0.497512x10^{-2}	0.497512x10^{-2}	0.497512x10^{-2}	0.497512x10^{-2}	0.497512x10^{-2}	0.497512x10^{-2}	0.497512x10^{-2}	0.497512x10^{-2}	0.497512x10^{-2}	0.497512x10^{-2}	0.497512x10^{-2}
C	-1.496264	-1.494843	-1.490563	-1.473536	-1.406024	-1.295804	-1.146198	-0.961784	-0.748132	-0.511752	-0.259826	0.000000
D	-0.497512	-0.496567	-0.493728	-0.482512	-0.439313	-0.373134	-0.291950	-0.205562	-0.124378	-0.581990x10^{-1}	-0.150000x10^{-1}	0.000000
E	-0.997509	-0.996561	-0.993708	-0.982357	-0.937349	-0.863869	-0.764132	-0.641189	-0.498755	-0.341168	-0.173217	0.000000
F	-0.498755	-0.498281	-0.496855	-0.491179	-0.468675	-0.431935	-0.382066	-0.320595	-0.249377	-0.170584	-0.866088x10^{-1}	0.000000
q	0.000000	-1.863148	-3.722878	-7.41714	-14.60875	-21.35657	-27.45559	-32.71998	-36.990698	-40.13712	-42.06434	-42.71315
Π_o	0.121643x10^{-1}	0.121643x10^{-1}	0.121643x10^{-1}	0.121643x10^{-1}	0.121643x10^{-1}	0.121643x10^{-1}	0.121643x10^{-1}	0.121643x10^{-1}	0.121643x10^{-1}	0.121643x10^{-1}	0.121643x10^{-1}	0.121643x10^{-1}
Ω_o	0.160501x10^{8}	0.160501x10^{8}	0.160501x10^{8}	0.160501x10^{8}	0.160501x10^{8}	0.160501x10^{8}	0.160501x10^{8}	0.160501x10^{8}	0.160501x10^{8}	0.160501x10^{8}	0.160501x10^{8}	0.160501x10^{8}
$\bar{F}$	0.900028x10^{-10}	0.900028x10^{-10}	0.900028x10^{-10}	0.900028x10^{-10}	0.900028x10^{-10}	0.900028x10^{-10}	0.900028x10^{-10}	0.900028x10^{-10}	0.900028x10^{-10}	0.900028x10^{-10}	0.900028x10^{-10}	0.900028x10^{-10}
$\sqrt{8\pi\Pi_o}$	0.552921	0.552921	0.552921	0.552921	0.552921	0.552921	0.552921	0.552921	0.552921	0.552921	0.552921	0.552921
K	-0.357089**	-0.366336	-0.384912	-0.429840	-0.558941	-0.759346	-1.063887	-1.531668	-2.295969	-3.755430	-7.917182	∞
β_y	0.705345x10^{-1}	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
β_z	0.705345x10^{-1}	0.705345x10^{-1}	0.705345x10^{-1}	0.705345x10^{-1}	0.705345x10^{-1}	0.705345x10^{-1}	0.705345x10^{-1}	0.705345x10^{-1}	0.705345x10^{-1}	0.705345x10^{-1}	0.705345x10^{-1}	0.705345x10^{-1}
$d\beta_y/d\tau$	0.000000	-0.131416	-0.262591	-0.523164	-1.030421	-1.506375	-1.936566	-2.307887	-2.609120	-2.831052	-2.966987	-3.01275
$d\beta_z/d\tau$	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

* These are computed from equations (29), (30) at $T = \tau = 0$.

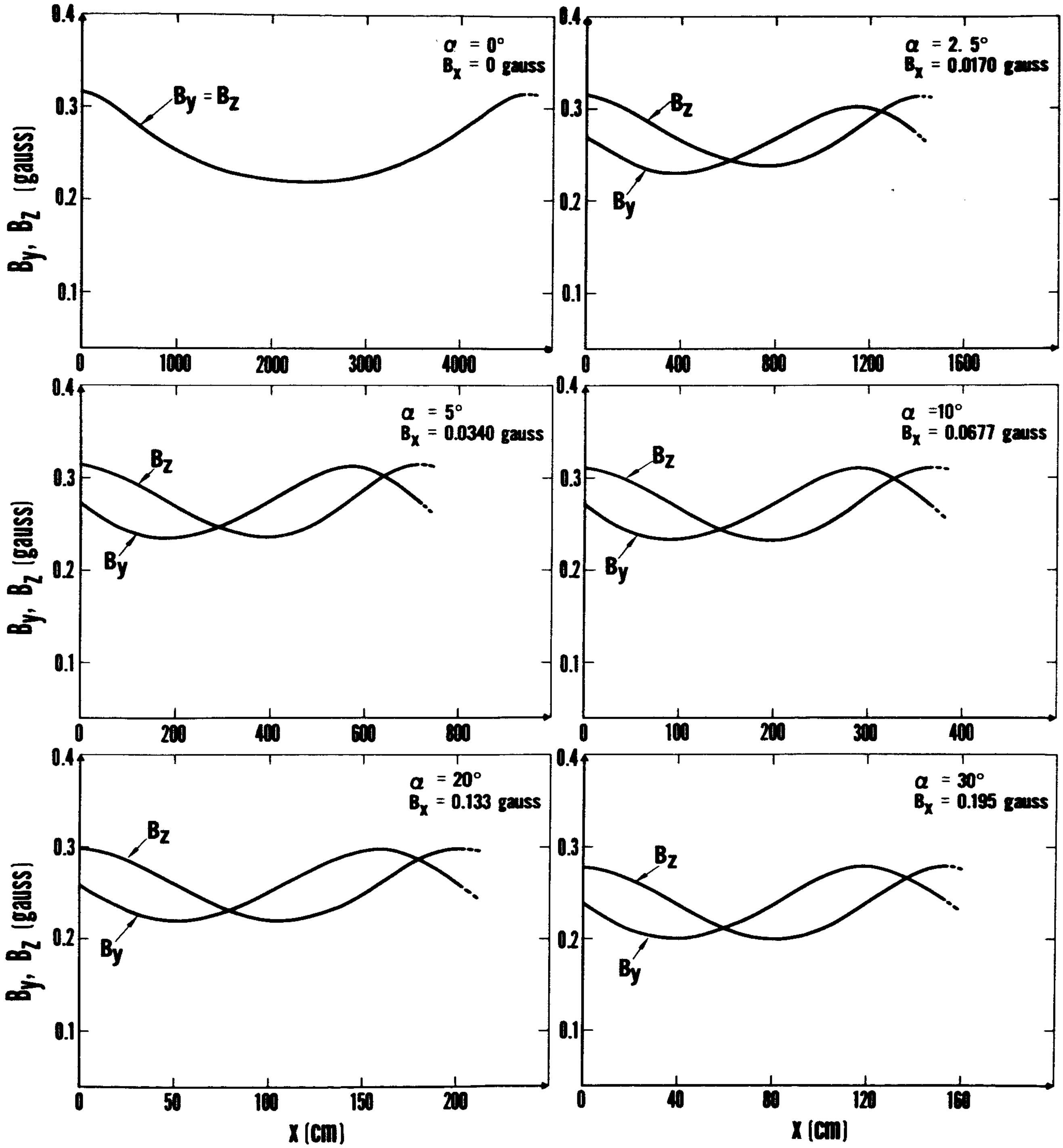
** Refer to part IV for these numbers.

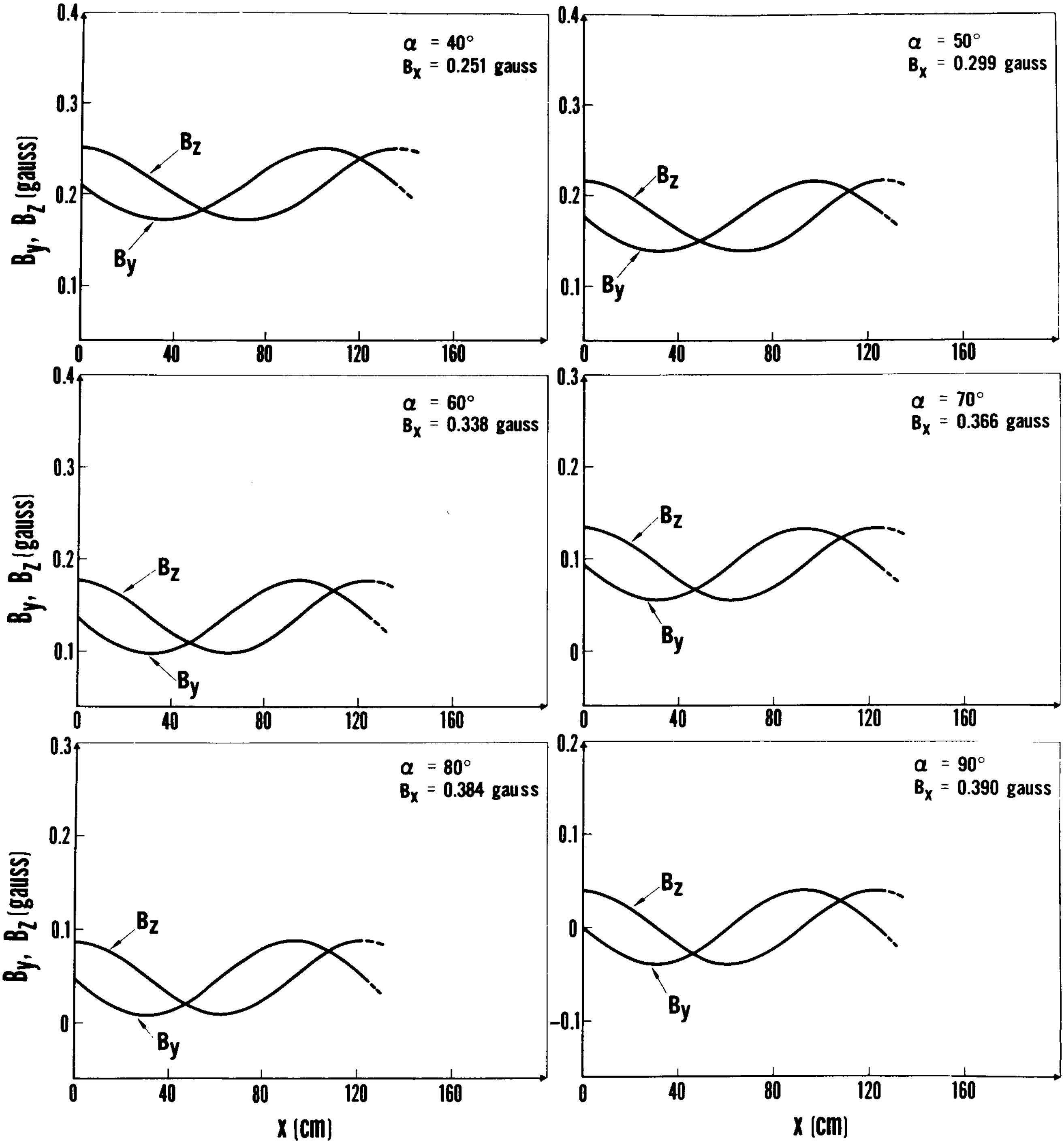
Other data: $N = n_p = n_e = 4 \times 10^5$ particles / cm^3.

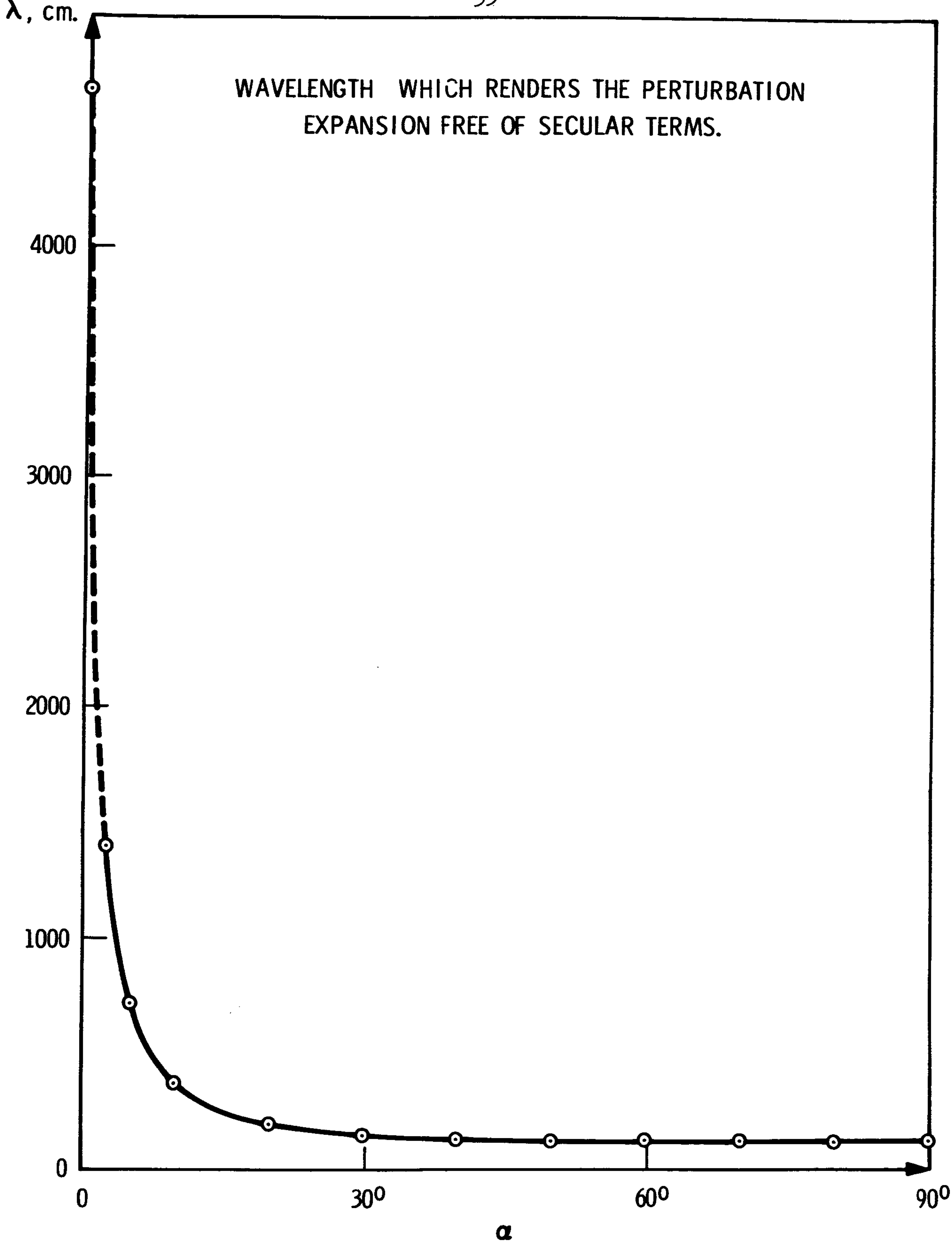
$m_p = 1.672 \times 10^{-24}$ gm.

$m_e = 9.108 \times 10^{-28}$ gm.

$e = 4.803 \times 10^{-10}$ e.s.u. $= 1.602 \times 10^{-20}$ e.m.u.


α = 0°
B_x = 0 gauss
$B_y = B_z$
α = 2.5°
B_x = 0.0170 gauss
α = 5°
B_x = 0.0340 gauss
α = 10°
B_x = 0.0677 gauss
α = 20°
B_x = 0.133 gauss
α = 30°
B_x = 0.195 gauss
B_z
B_y
B_y, B_z (gauss)
X (cm)
0.4
0.3
0.2
0.1
0
1000
2000
3000
4000
400
800
1200
1600
200
600
100
300
50
150
40
80
120
160


α = 40°
B_x = 0.251 gauss
α = 50°
B_x = 0.299 gauss
α = 60°
B_x = 0.338 gauss
α = 70°
B_x = 0.366 gauss
α = 80°
B_x = 0.384 gauss
α = 90°
B_x = 0.390 gauss
B_z
B_y
B_y, B_z (gauss)
x (cm)
0
40
80
120
160
0.4
0.3
0.2
0.1
0
-0.1


λ, cm.
WAVELENGTH WHICH RENDERS THE PERTURBATION EXPANSION FREE OF SECULAR TERMS.
4000
3000
2000
1000
0
0
30⁰
60⁰
90⁰
α

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